Machine Learning Statistical Gravity from Multi-Region Entanglement Entropy

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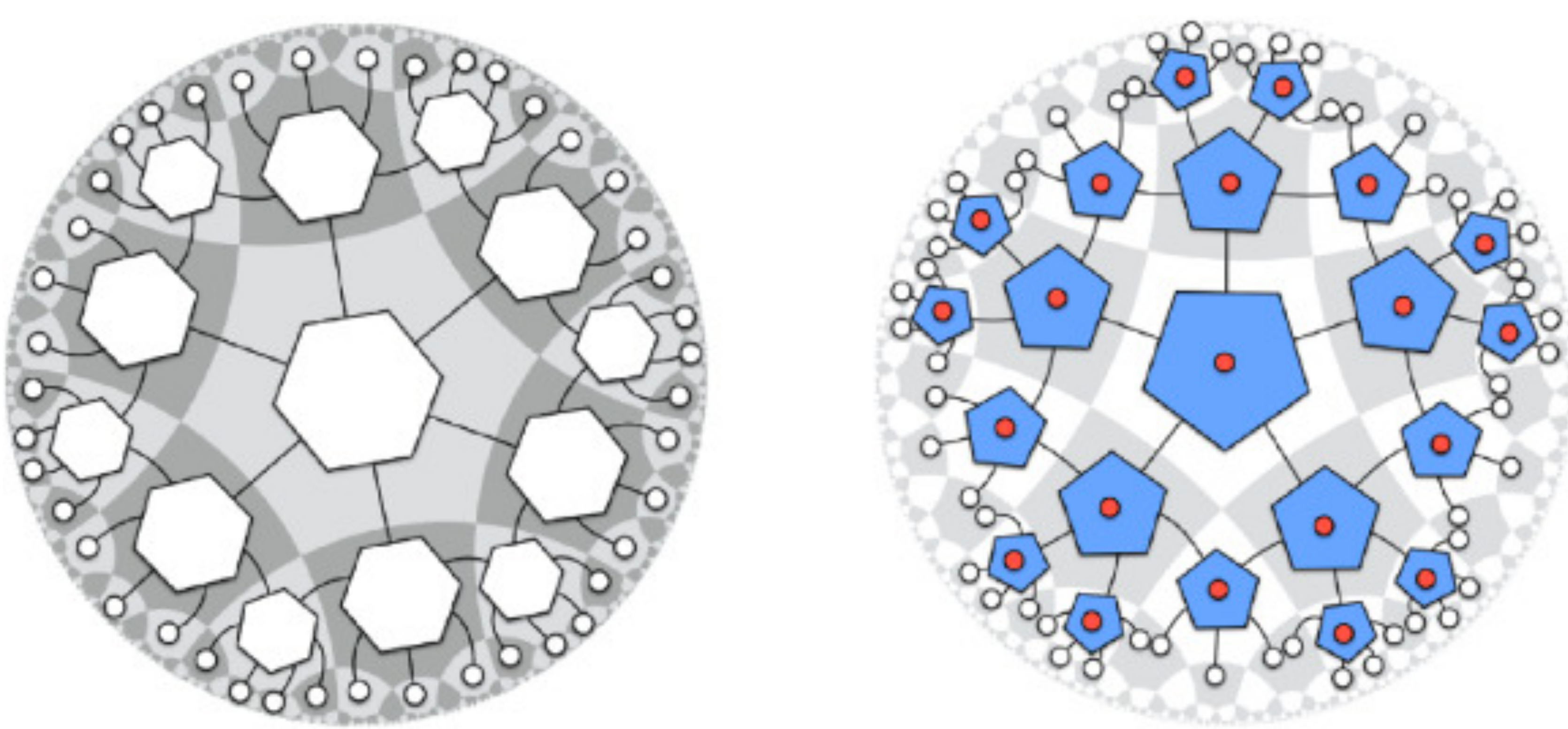
Holographic Reconstruction Problem

- Goal: reconstruct the holographic bulk geometry (gravity) from the holographic boundary data (quantum entanglement)
- Holographic duality Brown, Henneaux (1986), Witten (1998), Maldacena (1999) …
  - d-dimensional quantum field theory (boundary)
  - (d+1)-dimensional gravitational theory (bulk)
- Emergent spacetime from quantum entanglement

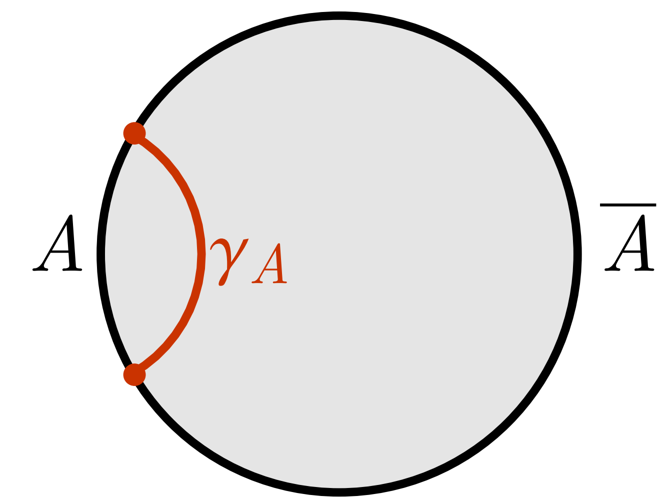


Pastawski, Yoshida, Harlow, Preskill (2015)

Holographic Reconstruction Problem

- Ryu-Takayanagi (RT) formula

$$S(A) = \frac{1}{4G_N} \min_{\gamma_A} |\gamma_A|$$



- In (1+1)D CFT, entanglement entropy

$$S(A) \sim c \log |A|$$

- Using AdS/CFT correspondence (in the classical gravity limit), geodesic (minimum surface) through the hyperbolic bulk

$$|\gamma_A| \sim \log |A|$$

- RT formula provides a geometric interpretation to quantum entanglement.

Holographic Reconstruction Problem

- Inverse problem: bulk reconstruction from boundary data
  - Geodesic lengths

    Porrati, Rabadan (2004); Hammersley (2006); Bilson (2008);
    Cao, Qi, Swingle, Tang (2020)

  - Extremal areas

    Bilson (2011); Alexakis, Balehowsky, Nachman (2017);
    Bao, Cao, Fischettti, Keeler (2019)

  - Entanglement data

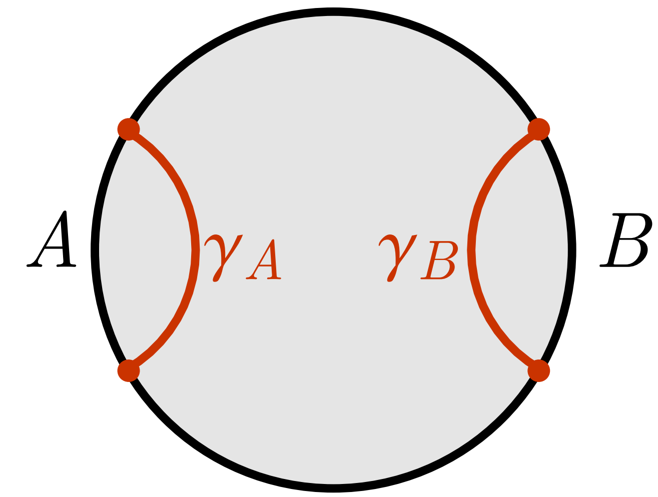
    You, Yang, Qi (2018); Roy, Sarkar (2018)

- Previous work: single-region entanglement → classical bulk geometry
- This work: multi-region entanglement → fluctuating bulk geometry (statistical gravity model)

Multi-Region Entanglement

- Mutual information

$$I(A:B) = S(A) + S(B) - S(AB)$$
$$\propto |\gamma_A| + |\gamma_B| - |\gamma_{A\cup B}|$$



  - For far separated regions *A* and *B*

$$|\gamma_{A\cup B}| = |\gamma_A| + |\gamma_B| \quad \Rightarrow \quad I(A:B) = 0$$

  - True for holographic CFTs
  - But not generally the case for many other quantum systems (e.g. free-fermion CFT)

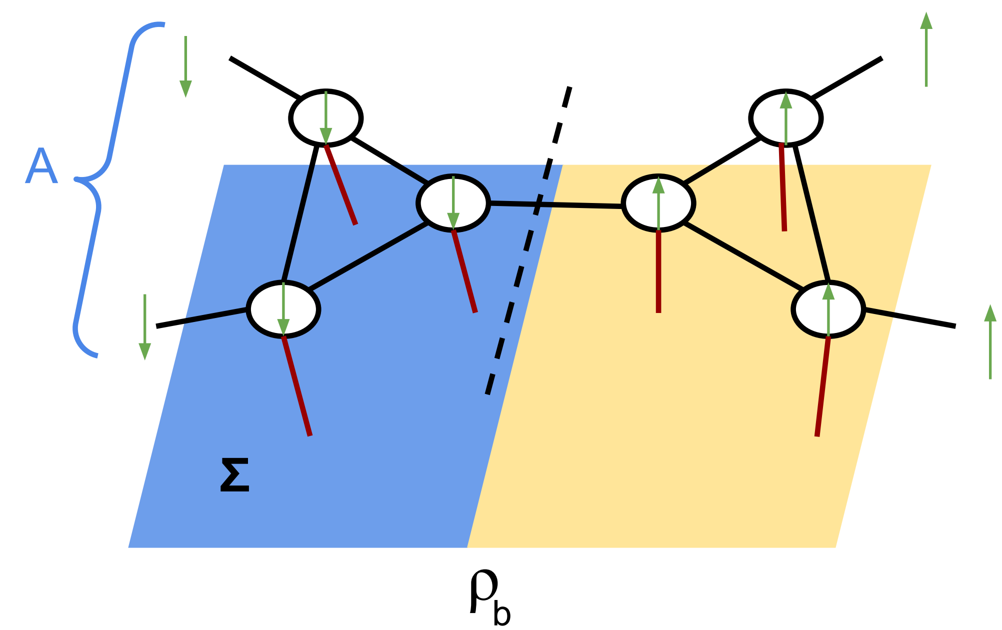
$$I(A:B) \propto d_{AB}^{-2}$$

- How to capture the non-vanishing mutual information?

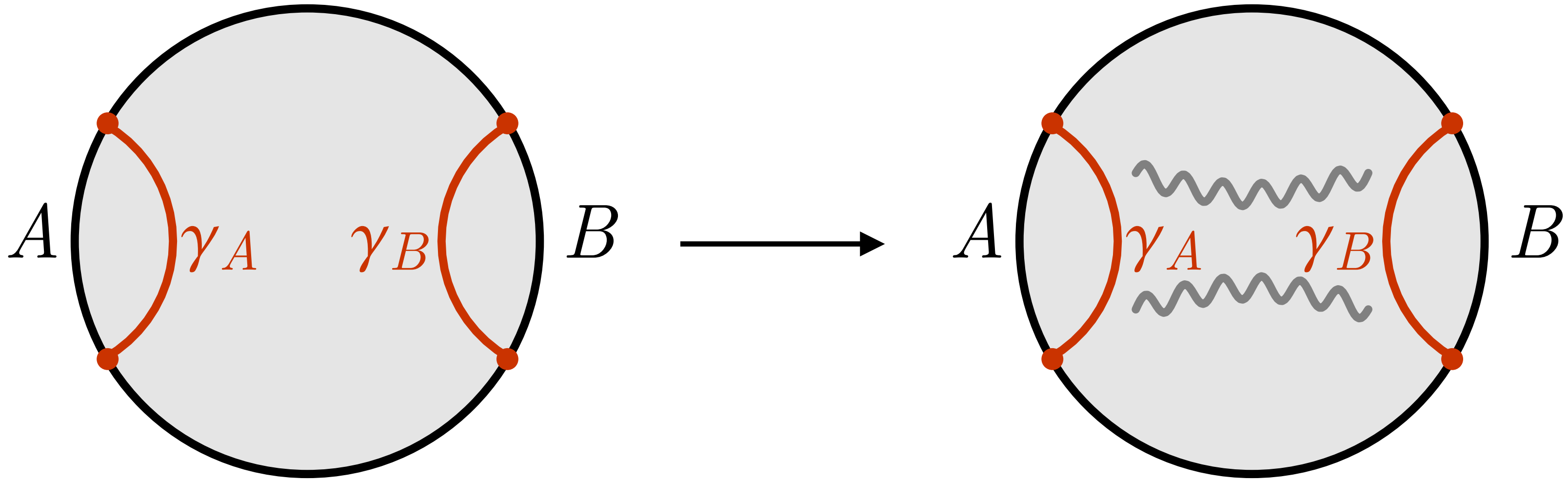
Multi-Region Entanglement

- Idea 1: introduce bulk mater field (QFT) to mediate the mutual information (MI)

  Faulkner, Lewkowycz, Maldacena (2013); Engelhardt, Wall (2015); Dong, Qi, Shangnan, Yang (2020)



- Idea 2: introduce gravitational fluctuations to mediate the MI (this work)

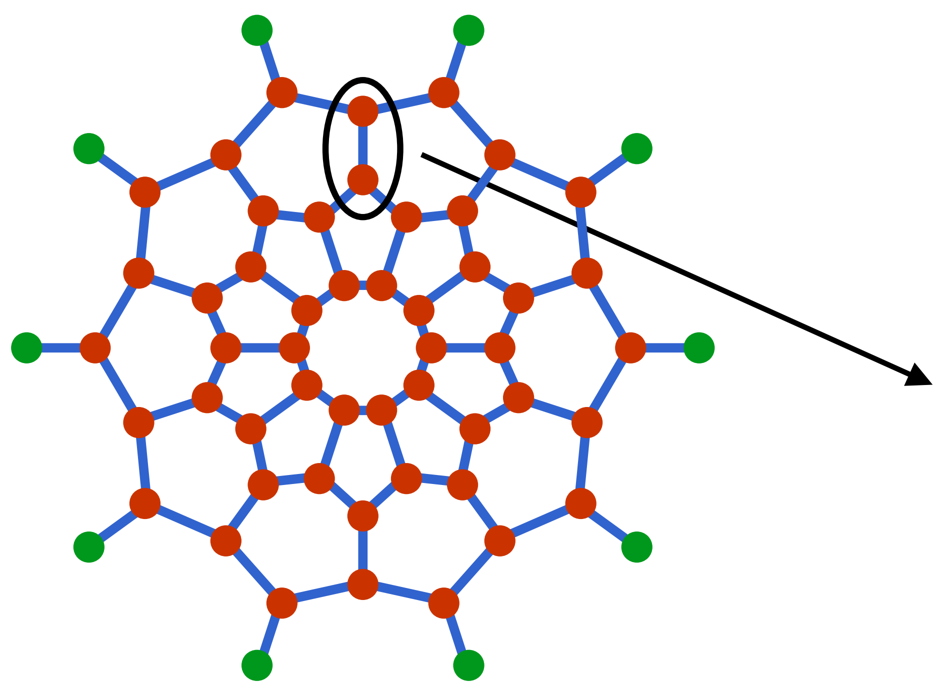


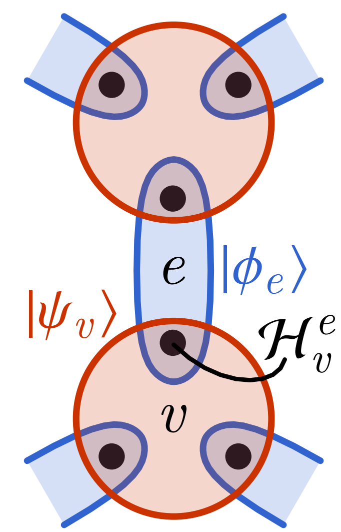
Use statistical learning to extract gravitational model from entanglement

Random Tensor Networks

- Framework: Random Tensor Network (RTN) holography
  - Tensor network: representation of quantum many-body state

Hayden, Nezami, Qi, Thomas, Walter, Yang (2016); Qi, Yang, You (2017)





  - Network geometry ~ holographic bulk geometry
  - RTN: random tensors + fixed bound-dimension
  - Reproduce RT formula in the large-bound-dimension limit

Random Tensor Networks

- Classical RTN holography
  - 2nd Renyi entropy of RTN state

$$e^{-S(A)} \propto \mathop{\mathbb{E}}_{|\Psi\rangle \in \mathcal{E}_{\mathrm{RTN}}} \mathrm{Tr}_A (\mathrm{Tr}_{\bar{A}} |\Psi\rangle\langle\Psi|)^2$$

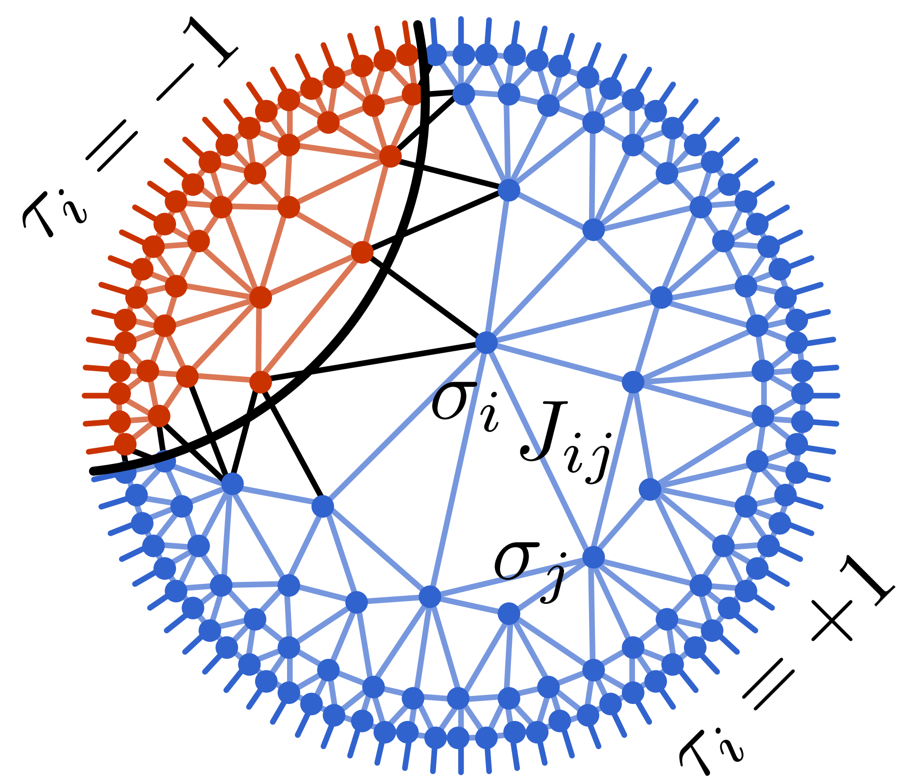
  - Average over random tensors

$$e^{-S(A)} \propto \sum_{[\sigma]} e^{-E[\sigma,\tau(A)]}$$

$$E[\sigma,\tau] = -\sum_{i,j} J_{ij}\sigma_i\sigma_j - h\sum_i \tau_i\sigma_i$$

subject to boundary condition

$$\tau_i(A) = \begin{cases} +1 & i \notin A, \\ -1 & i \in A. \end{cases}$$

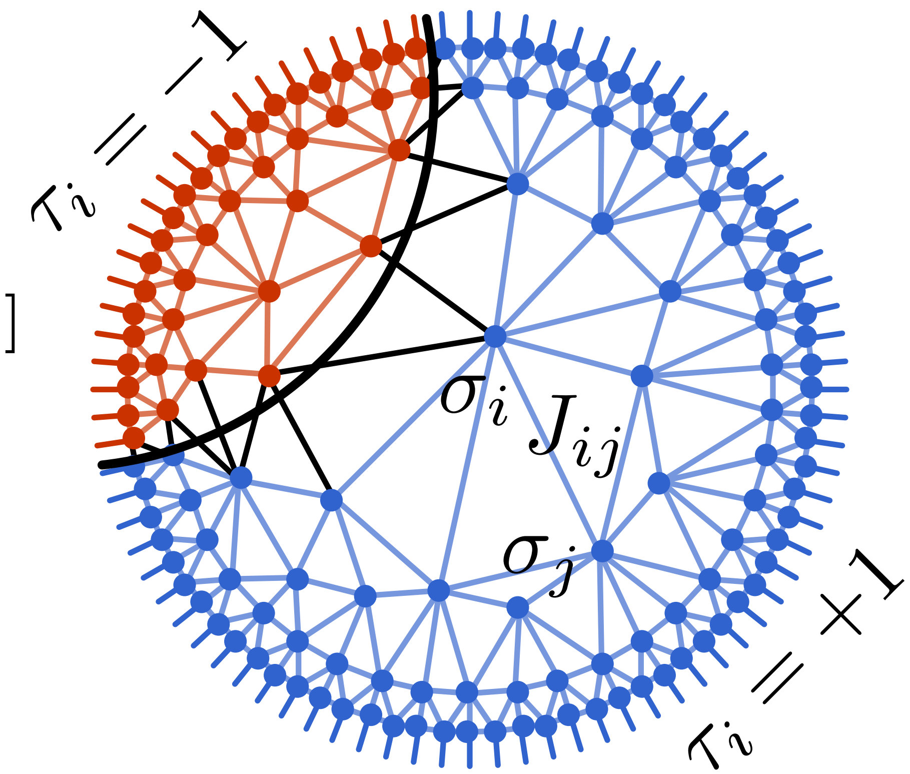


Random Tensor Networks

- Classical RTN holography
  - Average over random tensors

$$e^{-S(A)} \propto \sum_{[\sigma]} e^{-E[\sigma,\tau(A)]} = e^{-F[\tau(A)]}$$

$$E[\sigma,\tau] = -\sum_{i,j} J_{ij}\sigma_i\sigma_j - h\sum_i \tau_i\sigma_i$$



  - Entanglement entropy ~ Ising free energy
  - Ising couplings ~ log bond dimension

$$J_{ij}, h \propto \log D$$

  - Large bound dimension limit → deep in ferromagnetic phase → $F[\tau]$ dominated by shortest domain wall → geodesic distance through the bulk (RT formula)

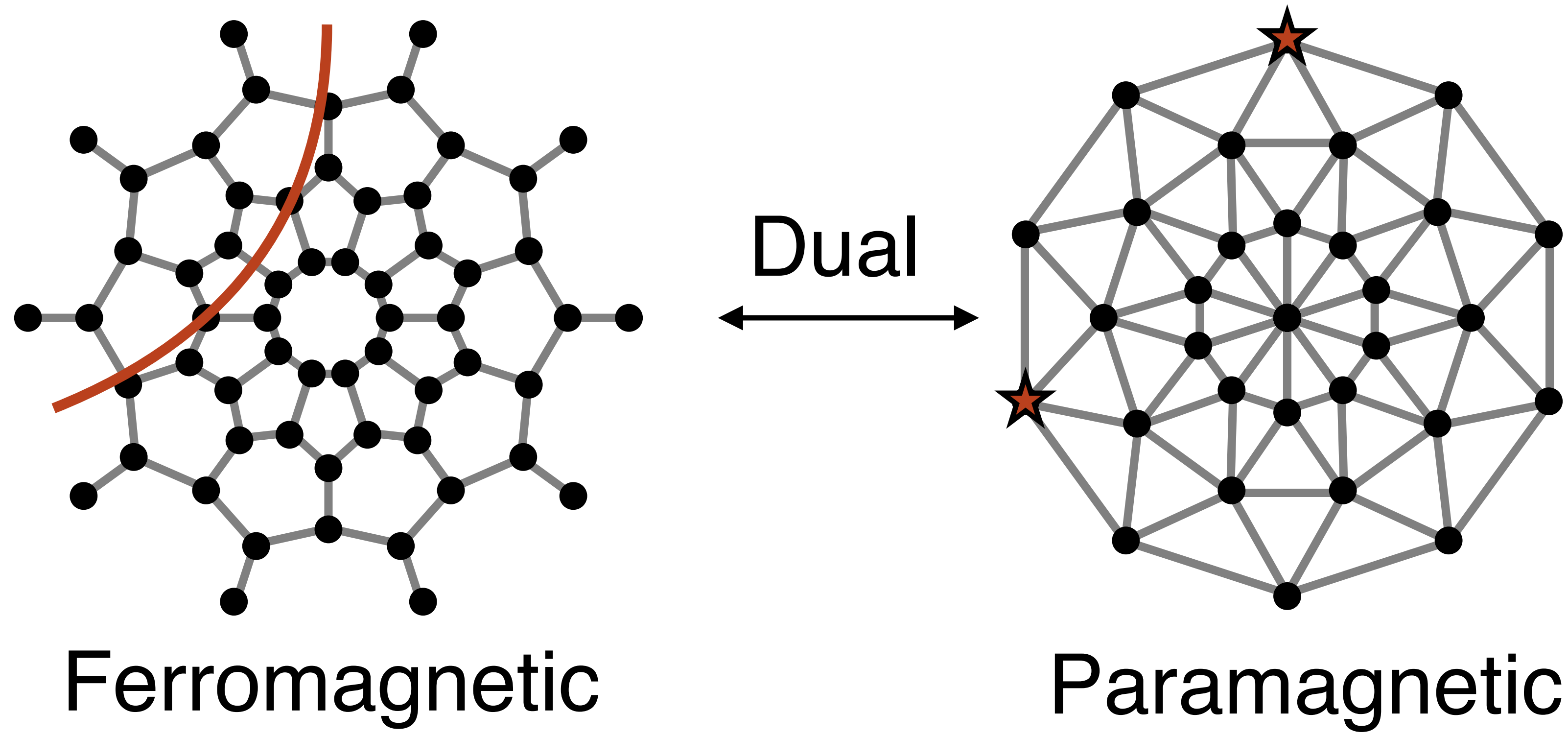
Fluctuating Random Tensor Networks

- Classical RTN
  - Fixed bond dimension → static geometry
  - Vanishing mutual information problem
- Fluctuating RTN
  - Random tensor + fluctuating bond dimensions → fluctuating geometry (gravity)
  - (Hopefully) better capture mutual information
  - Learns the underlying gravitational fluctuation from entanglement data

$$e^{-S(A)} \propto \sum_{[\sigma,J]} e^{-E[\sigma,\tau(A)|J]} \underbrace{P[J]}_{\text{Unknown joint distribution of Ising couplings (to learn)}}$$

Ising Duality

- Duality of planar-graph Ising model



$$e^{-S(A)} \propto e^{-F[\tau(A)]} \overset{\text{dual}}{=} \left\langle \prod_{i \in \partial A} \tilde{\sigma}_i \right\rangle$$ Correlation function of dual spins

- Dual spin: short range correlation $\langle \tilde{\sigma}_i \tilde{\sigma}_j \rangle \sim e^{-d_{ij}/\xi}$

$$S(A) \propto d_{ij} \propto |\gamma_A|$$ (RT formula)

Dual Ising Model

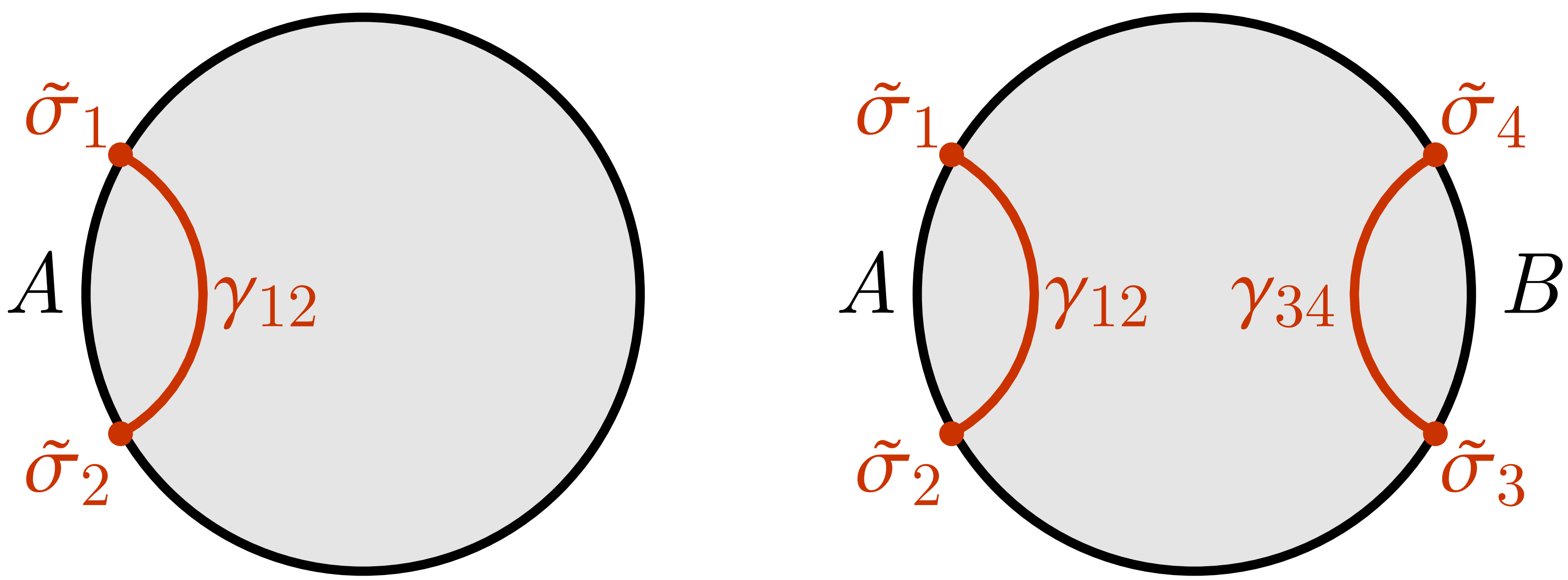
- Single-region entanglement ~ Two-point correlation

$$e^{-S(A)} \propto \langle \tilde{\sigma}_1 \tilde{\sigma}_2 \rangle \propto e^{-|\gamma_{12}|/\xi}$$

- Multi-region entanglement ~ multi-point correlation

$$e^{-S(AB)} \sim \langle \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\sigma}_3 \tilde{\sigma}_4 \rangle \sim e^{-|\gamma_{12}|/\xi} e^{-|\gamma_{34}|/\xi} \quad \text{(Classical)}$$

$$\Rightarrow S(AB) = S(A) + S(B) \Rightarrow I(A:B) = 0$$

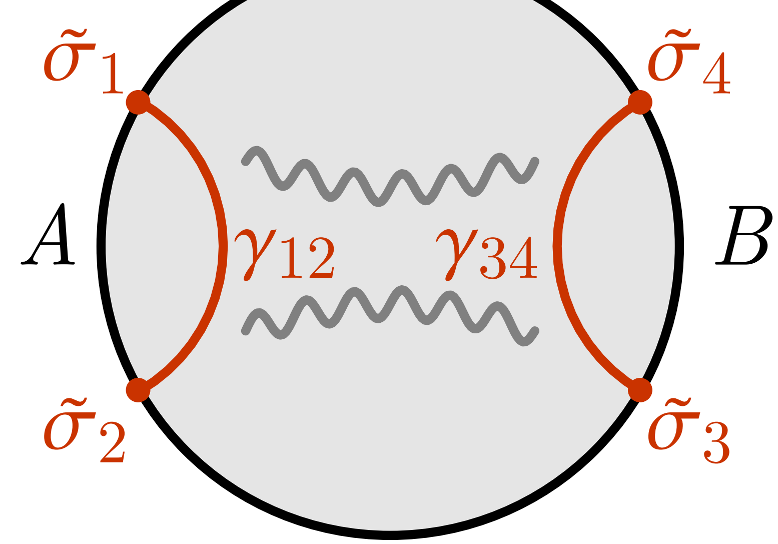


Fluctuating Dual Ising Model

- RTN: fluctuating bond dimension
  → (Dual) Ising model: fluctuating Ising coupling
  → Gravity model: fluctuating geodesic

$$
\begin{aligned}
e^{-S(AB)} &\sim \mathbb{E}\left(e^{-(|\gamma_{12}|+\delta|\gamma_{12}|)/\xi} e^{-(|\gamma_{34}|+\delta|\gamma_{34}|)/\xi}\right) \\
&\sim e^{-|\gamma_{12}|/\xi} e^{-|\gamma_{34}|/\xi} \; e^{\frac{1}{2\xi^2}\,\mathbb{E}\,\delta|\gamma_{12}|\delta|\gamma_{34}|} \\
&\sim e^{-S(A)} e^{-S(B)} e^{I(A:B)}.
\end{aligned}
$$

$$\Rightarrow I(A:B) \neq 0$$



- Mutual information mediated by gravitational fluctuation
- Conversely, gravity model can be extracted from multi-region entanglement

Massive Scalar Field

- Universality: paramagnetic Ising model is controlled by the massive scalar field fixed point $\tilde{\sigma}_i \to \phi(x)$

$$e^{-S(A)} \propto \sum_{[\phi,g]} \prod_{x \in \partial A} \phi(x) e^{-E[\phi|g]} P[g]$$

$$E[\phi|g] = \frac{1}{2} \int \mathrm{d}^2 x \sqrt{g} (g^{ij} \partial_i \phi \partial_j \phi + m^2 \phi^2)$$

- Simplification for 2D space,
  - Symmetric metric tensor g_{ij} has 3 components
  - 2 of them fixed by gauge $g_{ij} \to g_{ij} + \nabla_i \xi_j + \nabla_j \xi_i$
  - Remaining 1 parameterized by the Weyl field

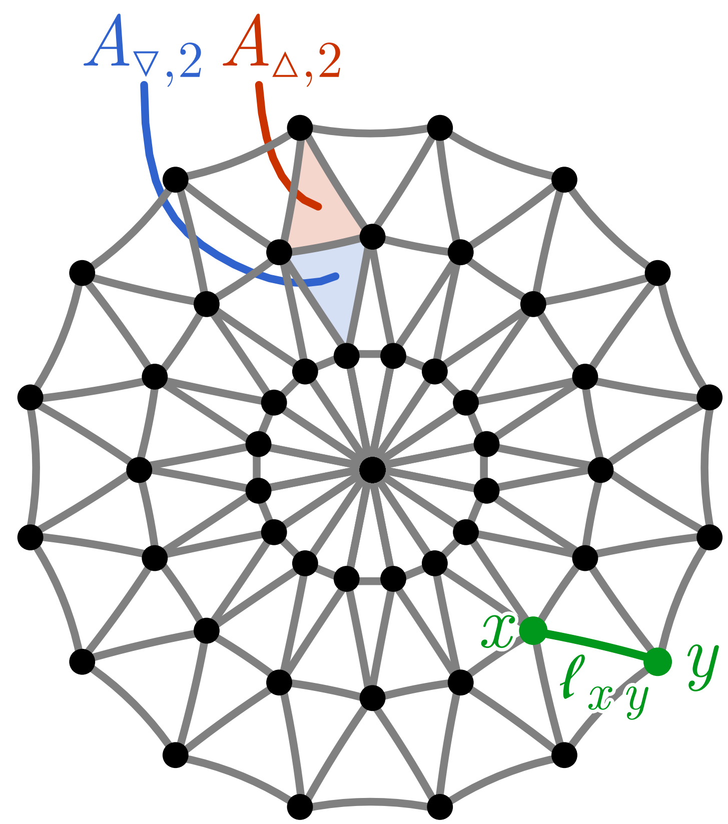
$$g_{ij}(x) = e^{2\omega(x)} \bar{g}_{ij}(x)$$

Massive Scalar Field

- Bulk model: massive scalar field coupled to Weyl field
  - Discretize using Regge calculus

$$E[\phi|\omega] = \sum_{\langle xy \rangle} \frac{A_{xy}}{2} \left(\frac{\phi_x - \phi_y}{\ell_{xy}} \right)^2 + \sum_x \frac{m^2 A_x}{2} e^{2\omega_x} \phi_x^2$$

$$\langle \textstyle\prod_{\partial A} \phi \rangle_\omega \equiv \frac{1}{Z[\omega]} \int_{[\phi]} \left(\prod_{x \in \partial A} \phi_x \right) e^{-E[\phi|\omega]}$$



  - Boundary: entanglement entropy

$$e^{-S(A)} = \int_{[\omega]} \langle \textstyle\prod_{\partial A} \phi \rangle_\omega P[\omega]$$

How to model the Weyl field fluctuation?

Generative Modeling

- Flow-based generative model
  - Trackable likelihood
  - Efficient sampling
  - Latent inference

Jimenez-Resende, Mohamed (2015);
Dinh, Sohl-Dickstein, Bengio (2016);
Kignma et. al. (2016) …

- Idea: use a trainable bijective map to deform a simple prior distribution to the target distribution

$$z \to \omega = G_\vartheta(z)$$

$$P_\vartheta[\omega] = P(z)\det\left(\frac{\partial G_\vartheta(z)}{\partial z}\right)^{-1}$$

  - Generator G_ϑ can be model by neural networks made of bijective units
  - Designed to preserve bulk translation/reflection symm.

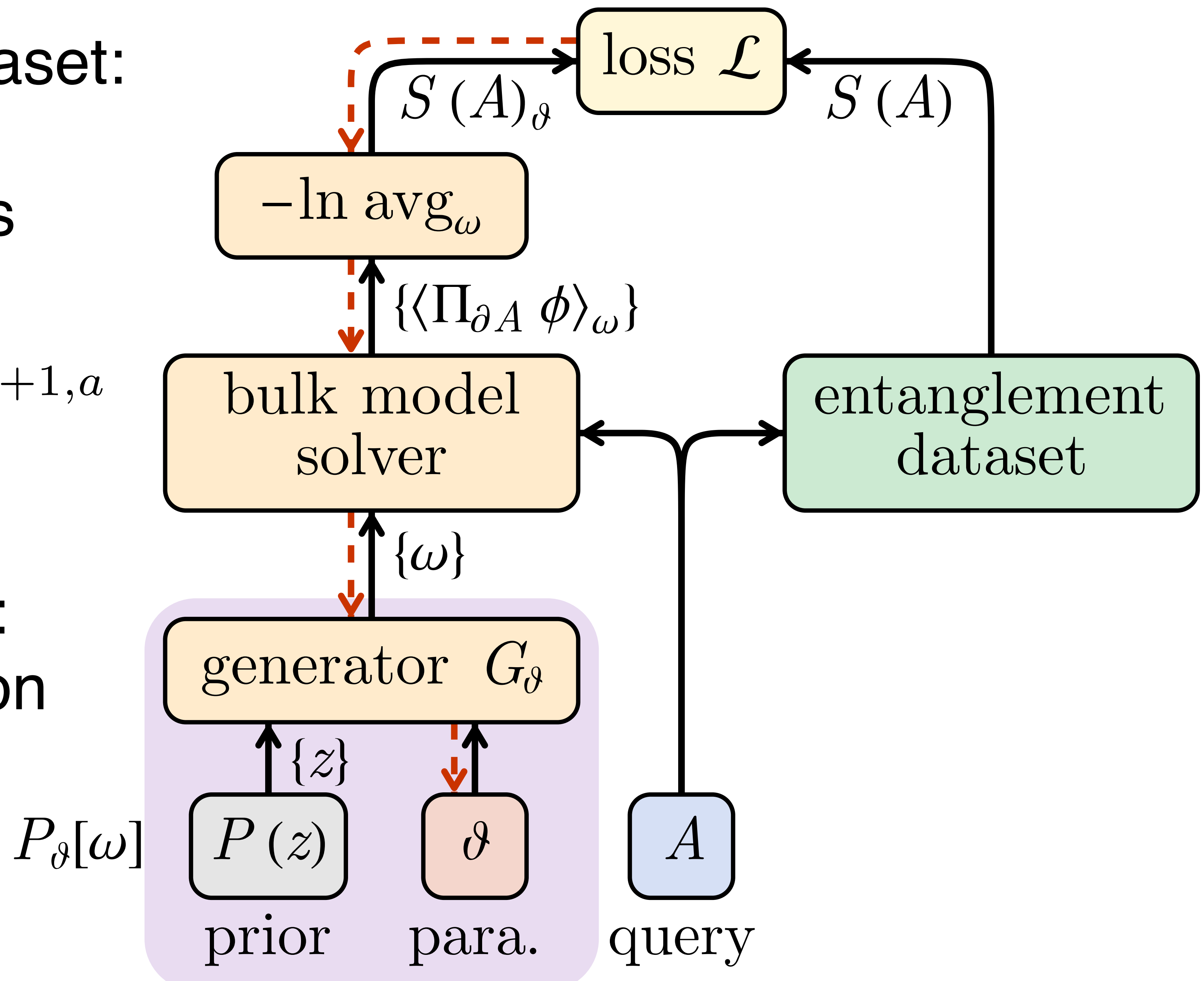
Generative Modeling

- Loss Function $\mathcal{L}_\vartheta = \underset{A}{\mathrm{avg}}\left(1 - e^{S(A)_\vartheta - S(A)}\right)^2$

- Entanglement dataset: (1+1)D massless Majorana fermions

$$H = \sum_{a=1}^{N} \sum_{j} \mathrm{i}\chi_{j,a}\chi_{j+1,a}$$

- Bulk model solver: compute correlation given Weyl field background



Fitting Entanglement Data

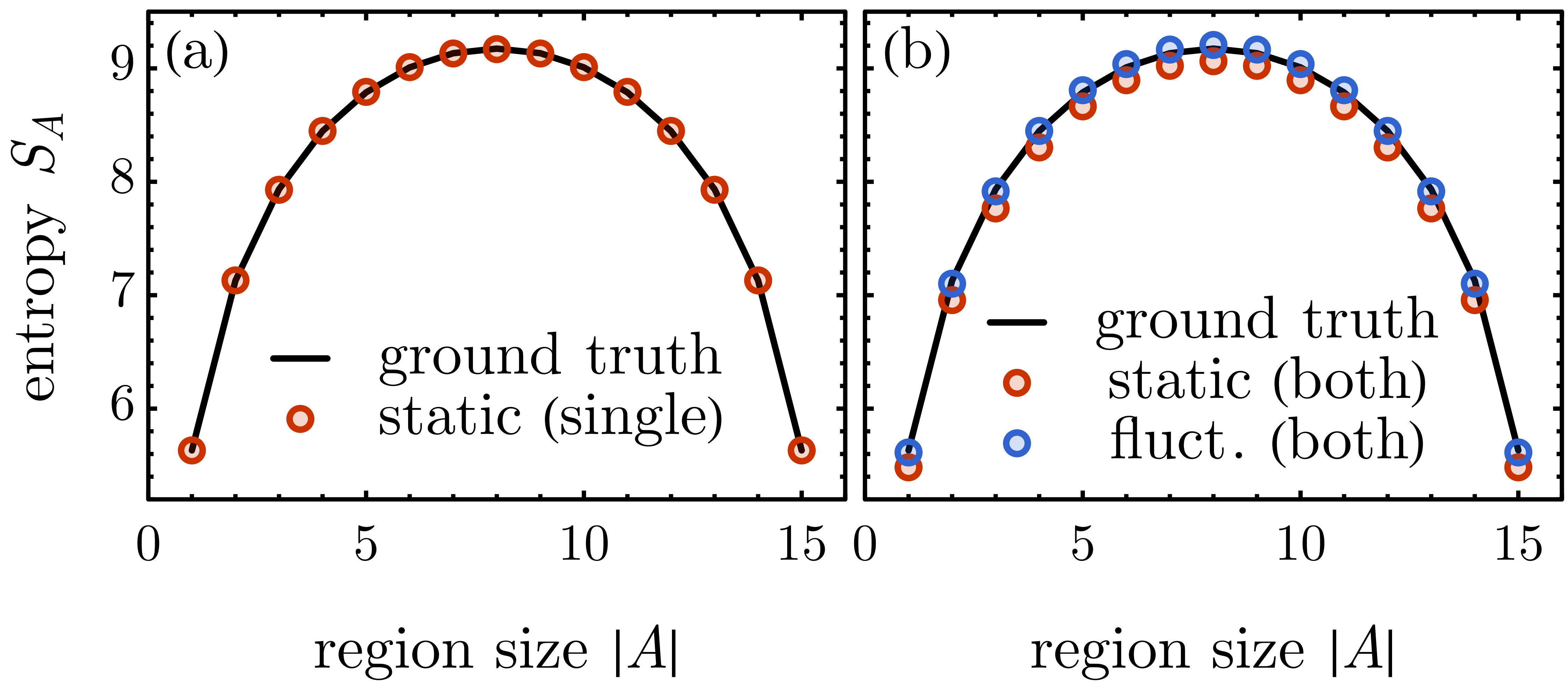
- Entanglement data
  - Collected from free fermion chain (32 sites) with large central charge c = 8
  - Training data: 1-, 2-region entanglement
  - Test data: 3-region entanglement

	Model Training set	static single	static single+double	fluctuating single+double
Test set	single	8.7×10^{-6}	2.1×10^{-2}	1.5×10^{-3}
	double	1.1×10^{-1}	3.9×10^{-2}	5.7×10^{-3}
	triple	7.5×10^{-1}	6.0×10^{-1}	3.1×10^{-1}

Final loss function (relative error in $e^{-S(A)}$)

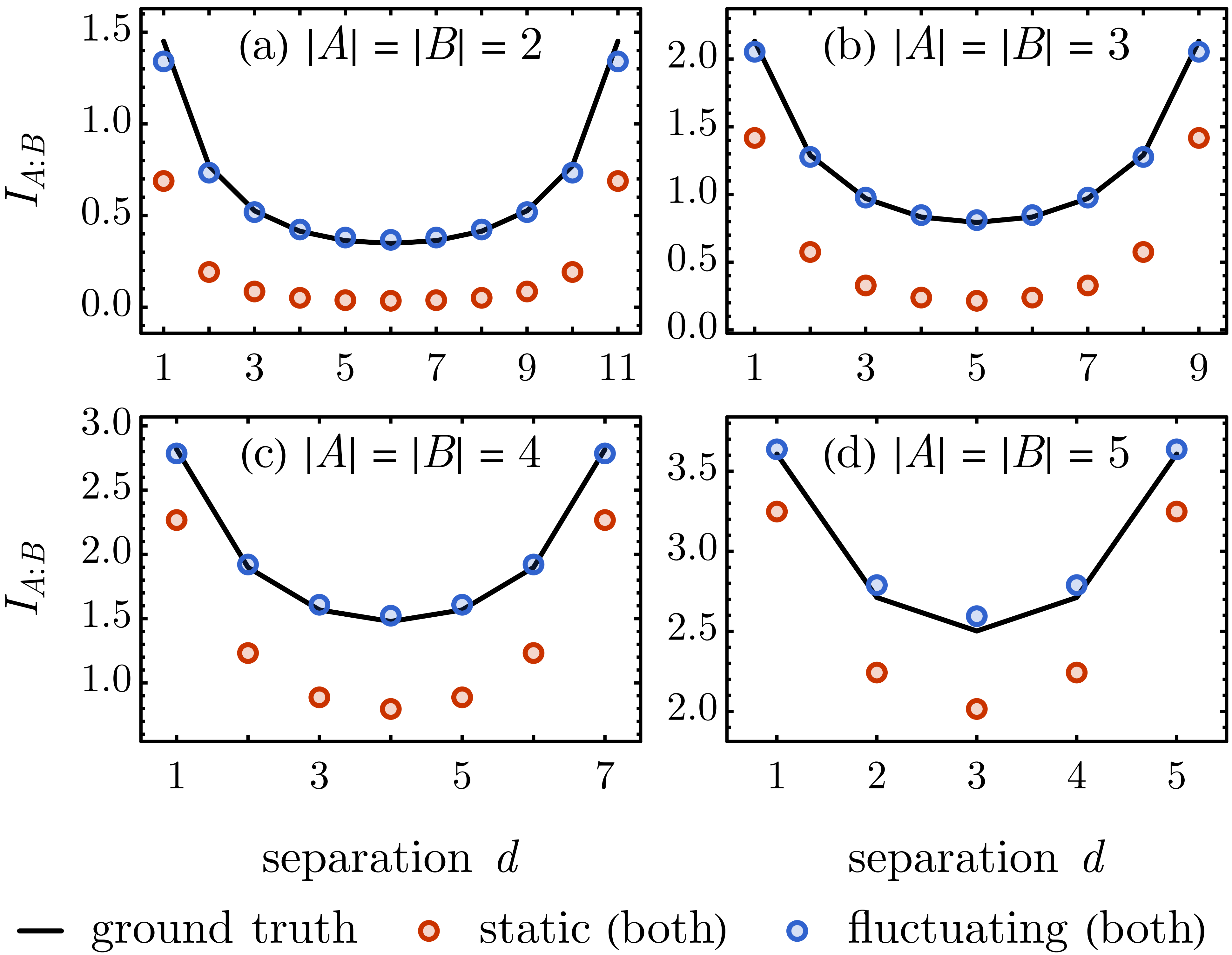
Fitting Entanglement Data

- Predicting Single-Region Entropy



Fitting Entanglement Data

- Predicting Mutual Information

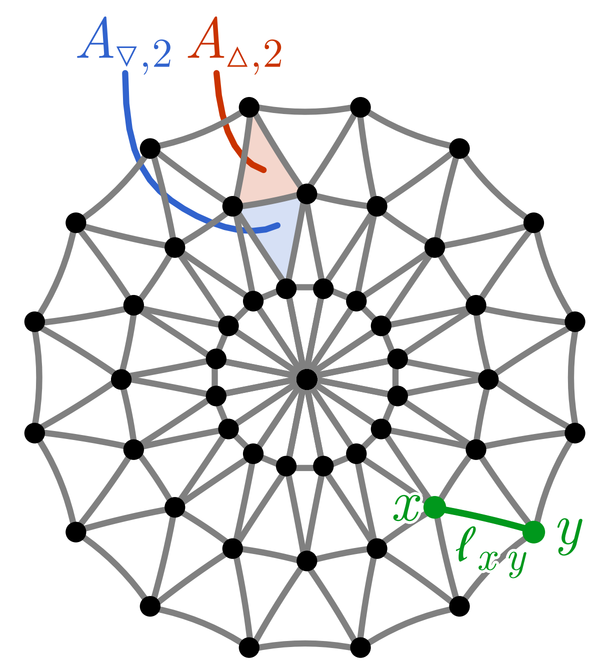


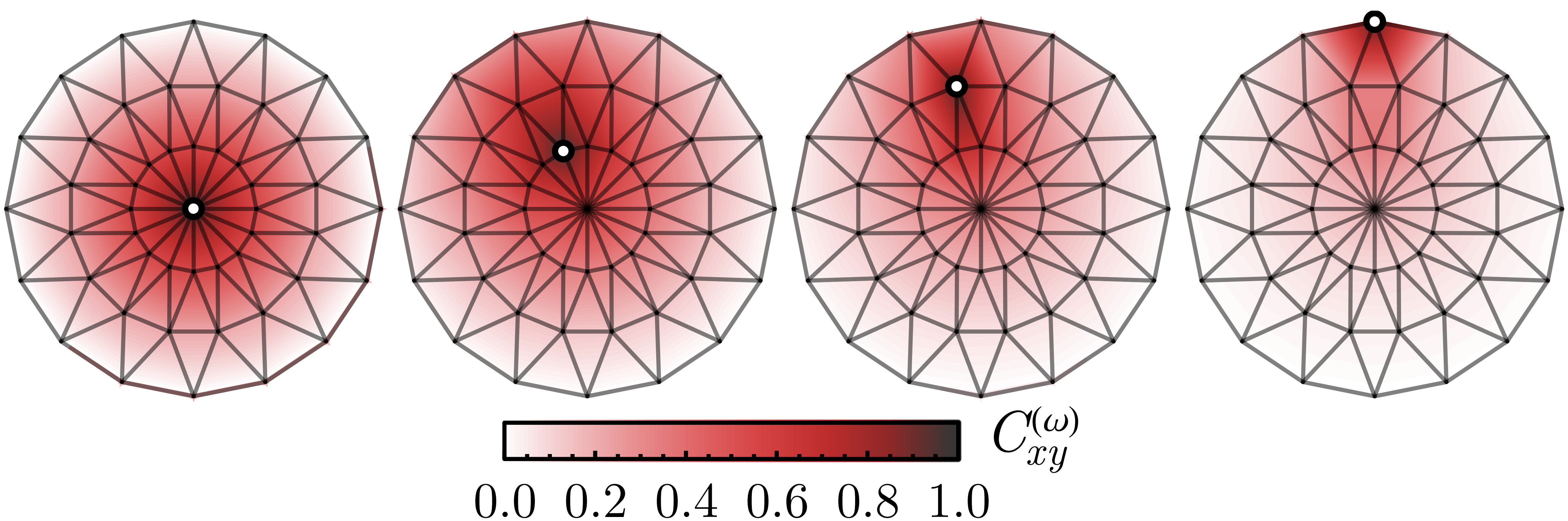
Weyl Field Correlation

- Define Weyl field correlation

$$C_{xy}^{(\omega)} = \frac{\Sigma_{xy}^{(\omega)}}{\sqrt{\Sigma_{xx}^{(\omega)}\Sigma_{yy}^{(\omega)}}}$$

$$\Sigma_{xy}^{(\omega)} = \mathop{\mathbb{E}}_{[\omega]\sim P_\vartheta} \omega_x \omega_y = \int_{[\omega]} P_\vartheta[\omega]\omega_x\omega_y$$



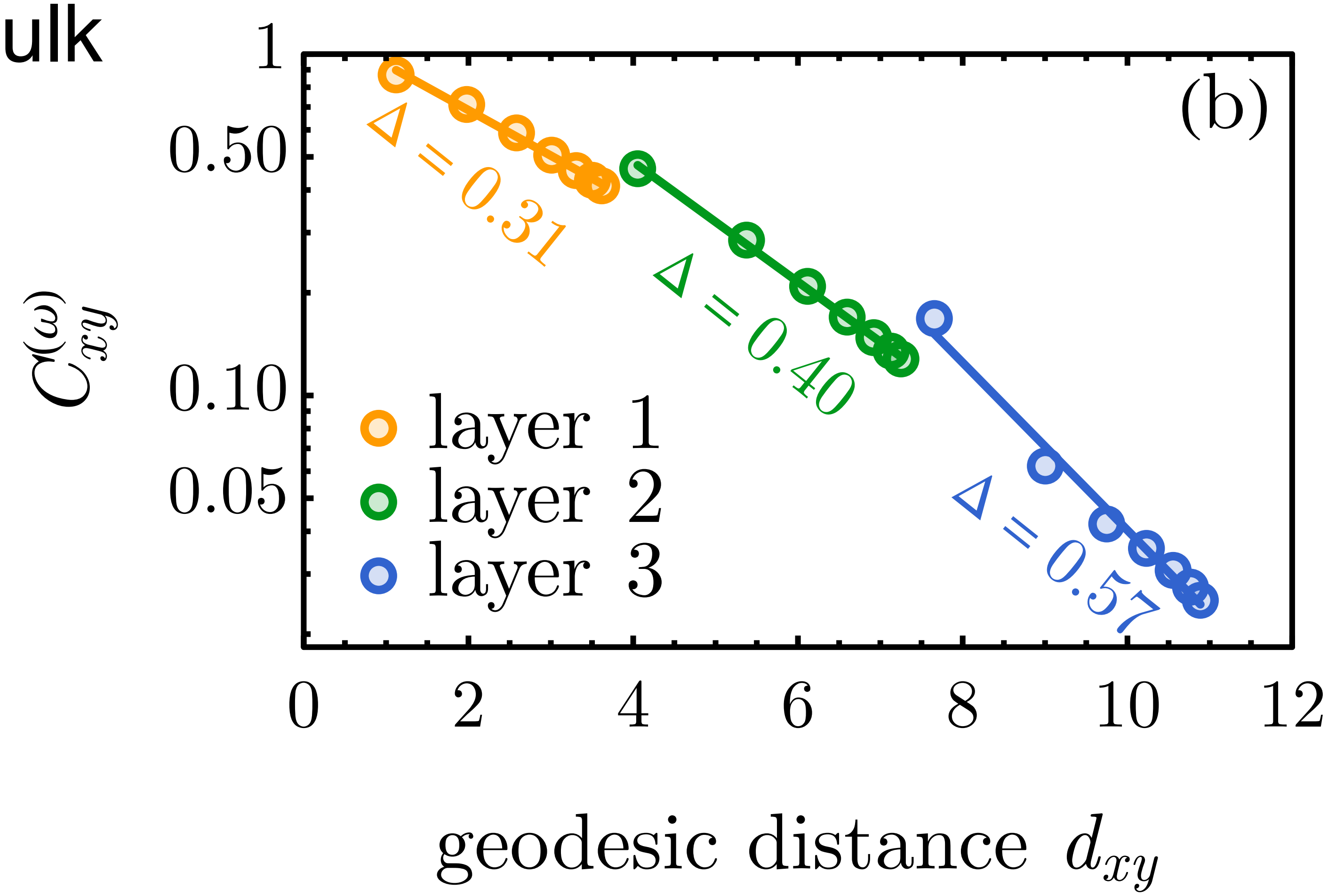


Weyl Field Correlation

- Emergent locality: Weyl field correlation decays exponentially in the bulk (while the locality was not imposed in the generative model)

$$C_{xy}^{(\omega)} \sim \exp(-\Delta d_{xy})$$

- Inverse correlation length Δ (scaling dim.) almost universal in the bulk

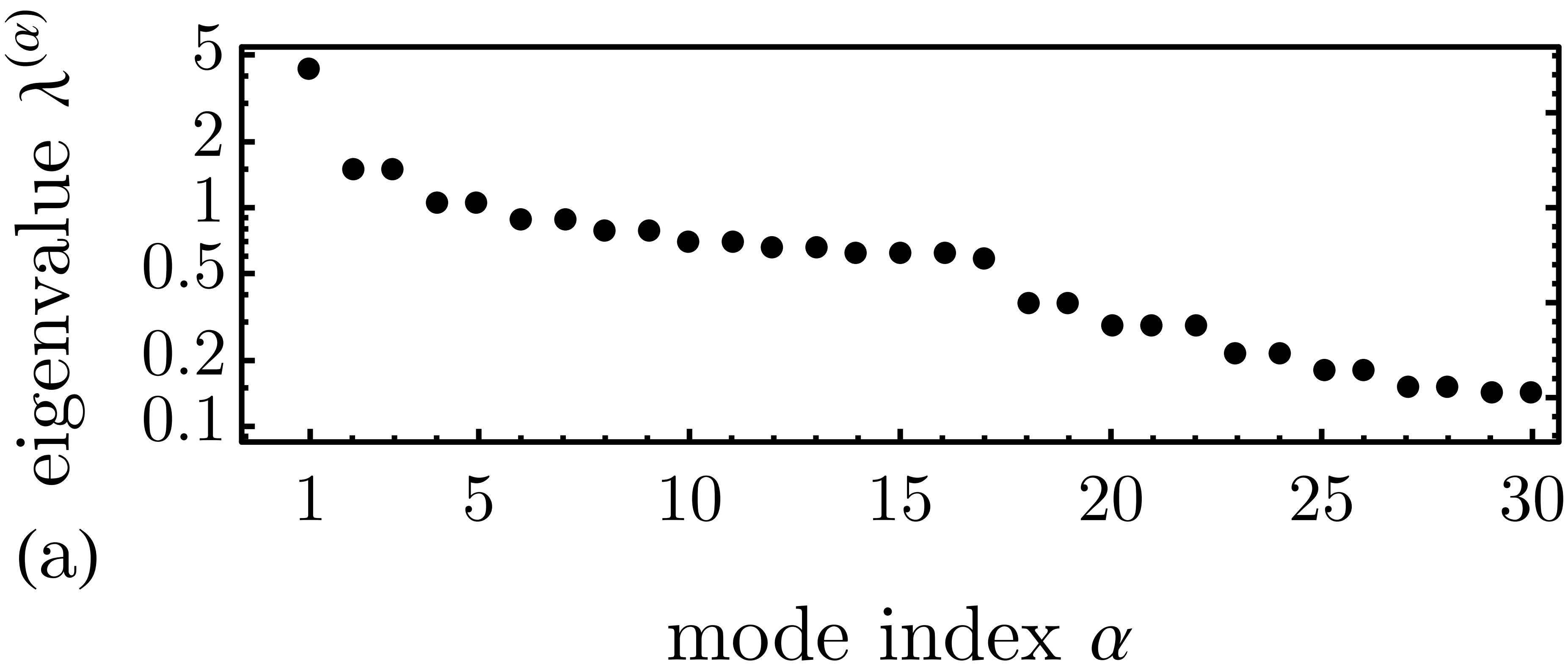


Gravitational Fluctuations

- Spectral decomposition of the Weyl field covariance

$$\Sigma_{xy}^{(\omega)} = \sum_{\alpha} \lambda^{(\alpha)} \omega_x^{(\alpha)} \omega_y^{(\alpha)}$$

  - Spectrum



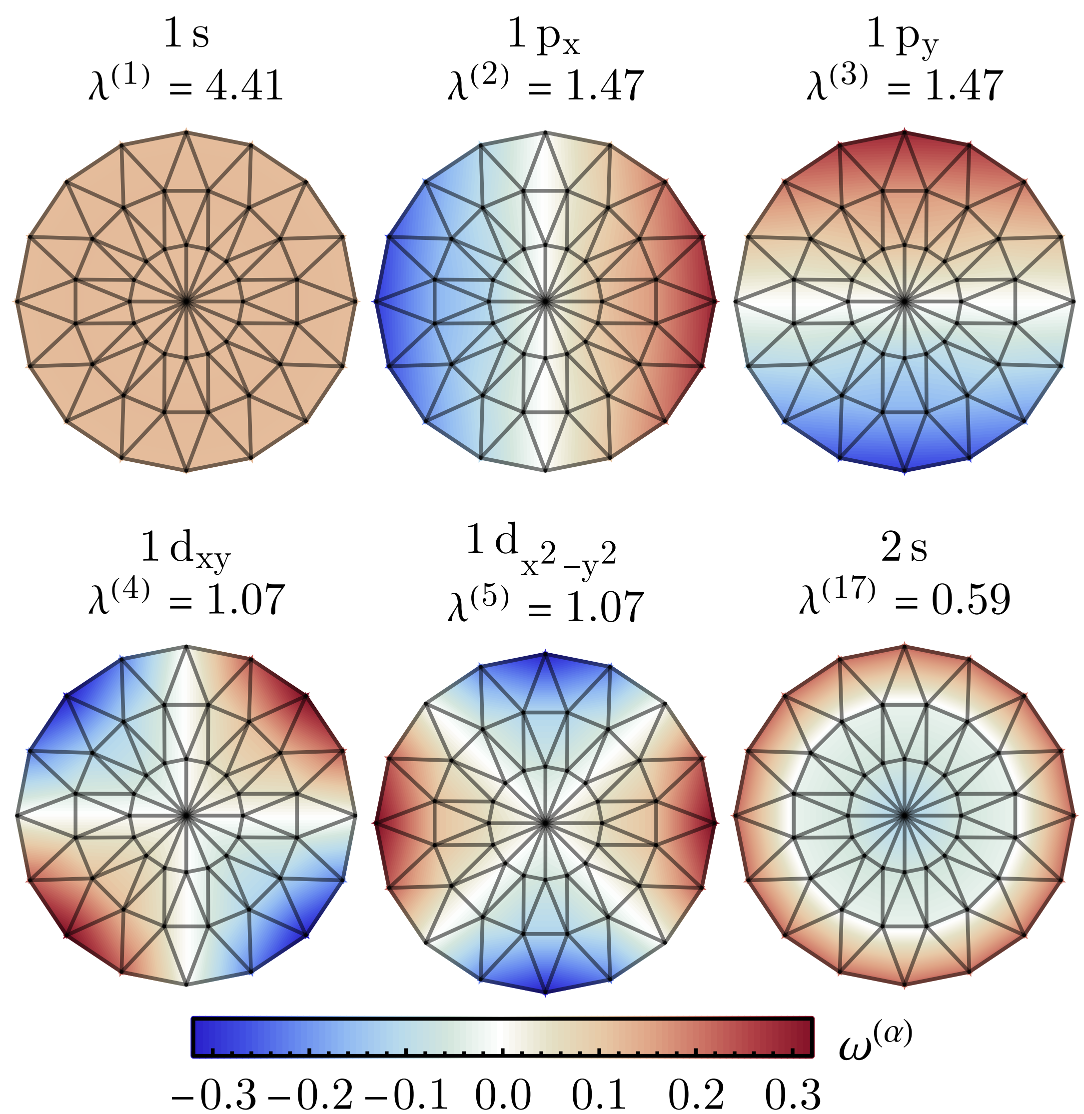
(a)

Gravitational Fluctuations

- Spectral decomposition of the Weyl field covariance

$$\Sigma_{xy}^{(\omega)} = \sum_{\alpha} \lambda^{(\alpha)} \omega_x^{(\alpha)} \omega_y^{(\alpha)}$$

  - Leading fluctuation modes



Gravitational Fluctuations

- Spectral decomposition of the Weyl field covariance

$$\Sigma_{xy}^{(\omega)} = \sum_{\alpha} \lambda^{(\alpha)} \omega_x^{(\alpha)} \omega_y^{(\alpha)}$$

- Effective gravitational action (Gaussian level)

$$S[\omega] = \frac{1}{2} \sum_{x,y} \omega_x (\Sigma^{(\omega)})^{-1}_{xy} \omega_y + \cdots$$

  such that $P[\omega] \propto e^{-S[\omega]}$

- Conclusion:
  - Learning to fit multi-region entanglement
  - Model learns an underlying statistical gravity model (in terms of the Weyl field)
  - Data-driven approach to establish holographic duality

Matter Field Mass Renormalization

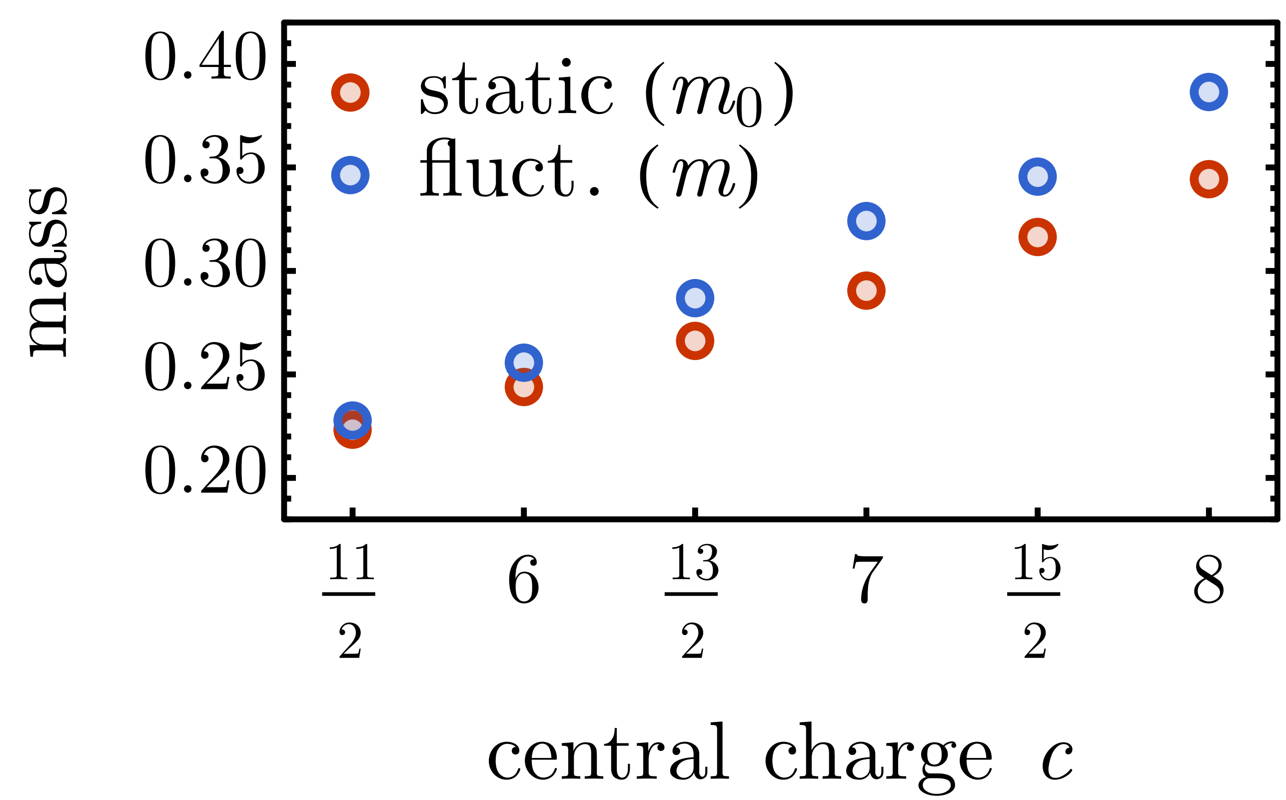
- To predict single-region entanglement
  - On static background geometry

$$e^{-S_A} \sim e^{-m_0|\gamma_A|}$$

  - With gravitational fluctuation (geodesic fluctuates)

$$e^{-S_A} \sim \mathop{\mathbb{E}}_{\omega} e^{-m(|\gamma_A|+\delta|\gamma_A|)} \simeq e^{-m(|\gamma_A|-\frac{m}{2}\mathbb{E}_\omega(\delta|\gamma_A|)^2)}$$

  - Matter field mass must be renormalized as $m > m_0$



Summary

- A machine-learning approach to extract holographic statistical gravity theory from multi-region entanglement data.
  - Generalize the RTN holography to include bond dimension fluctuations
  - Data-driven discovery of bulk gravity model with emergent locality
  - Practical use: a predictive model for quantum many-body entanglement structure → quantum circuit optimization / quantum algorithm design

- Data-driven discovery of physics theory from data?
  - Generative modeling → action
  - Neural ODE → equation of motion